

True and fake Lax pairs: how to distinguish them

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Abstract

The gauge-invariant description of zero-curvature representations of evolution equations is applied to the problem of how to distinguish the fake Lax pairs from the true Lax pairs. The main difference between the true Lax pairs and the fake ones is found in the structure of their cyclic bases.

In this note, we use the gauge-invariant description of zero-curvature representations of evolution equations, which was developed in [1], to solve the problem of how to distinguish the fake Lax pairs, introduced in [2], from the true Lax pairs. This approach is algorithmic and exploits only the x -part of a studied Lax pair. The main difference between the fake Lax pairs and the true ones turns out to be clearly seen from the structure of their *cyclic bases* (we use the terminology of [1] and explain it step by step in what follows).

Let us study a true Lax pair first. We take the true Lax pair of the modified KdV equation $u_t + u_{xxx} - 6u^2u_x = 0$ from [2]:

$$\psi_{xx} = \left(u^2 + ik \frac{u_x}{u} - k^2 \right) \psi + \frac{u_x}{u} \psi_x, \quad (1)$$

$$\psi_t = \left(-ik \frac{u_{xx}}{u} - 2k^2 \frac{u_x}{u} \right) \psi + \left(-\frac{u_{xx}}{u} + 2u^2 + 2ik \frac{u_x}{u} + 4k^2 \right) \psi_x, \quad (2)$$

where k is an essential (spectral) parameter. Introducing the vector Φ ,

$$\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \quad \phi_1 = \psi, \quad \phi_2 = \psi_x, \quad (3)$$

we rewrite the Lax pair (1)–(2) as

$$\Phi_x = X\Phi, \quad \Phi_t = T\Phi, \quad (4)$$

where

$$X = \begin{pmatrix} 0 & 1 \\ u^2 + ik u_x/u - k^2 & u_x/u \end{pmatrix} \quad (5)$$

(the explicit form of the matrix T will not be used in what follows). The compatibility condition

$$D_t X = D_x T - [X, T] \quad (6)$$

of the over-determined linear system (4) is the *zero-curvature representation* of the modified KdV equation (D_t and D_x stand for the total derivatives, the square brackets denote the commutator).

For a generic matrix $X(x, u, u_x, \dots, u_{x\dots x})$ and a generic evolution equation $u_t = h(x, u, u_x, \dots, u_{x\dots x})$, we can rewrite the zero-curvature representation (6) in its *characteristic form* [1]:

$$h C_u = \nabla H, \quad (7)$$

where ∇ is defined by $\nabla M = D_x M - [X, M]$ for any matrix M ; the matrix H is determined by X and T ; and the matrix C_u is the *characteristic* of X with respect to u , defined as

$$C_u = \frac{\partial X}{\partial u} - \nabla \left(\frac{\partial X}{\partial u_x} \right) + \nabla^2 \left(\frac{\partial X}{\partial u_{xx}} \right) - \nabla^3 \left(\frac{\partial X}{\partial u_{xxx}} \right) + \dots \quad (8)$$

Under the *gauge transformation*

$$X' = S X S^{-1} + (D_x S) S^{-1}, \quad T' = S T S^{-1} + (D_t S) S^{-1}, \quad (9)$$

generated by the transformation $\Phi' = S \Phi$ with any matrix S ($\det S \neq 0$), the corresponding transformation of (7) is tensor: $C'_u = S C_u S^{-1}$, $\nabla' = S \nabla S^{-1}$, $H' = S H S^{-1}$. The *cyclic basis* is the sequence of linearly independent matrices $C_u, \nabla C_u, \dots, \nabla^{n-1} C_u$, where n is maximal. In the *closure equation* of the cyclic basis,

$$\nabla^n C_u = a_0 C_u + \dots + a_{n-1} \nabla^{n-1} C_u, \quad (10)$$

the coefficients a_0, \dots, a_{n-1} (and, of course, the order n) are invariants with respect to the transformation (9).

Returning to the particular matrix X (5), we compute C_u and $\nabla^i C_u$, $i = 1, 2, 3$, and find that $C_u, \nabla C_u$ and $\nabla^2 C_u$ are linearly independent, whereas

$$\nabla^3 C_u = 4k^2 \frac{u_x}{u} C_u + 4(u^2 - k^2) \nabla C_u + \frac{u_x}{u} \nabla^2 C_u. \quad (11)$$

We see that, in the case of the true Lax pair of the modified KdV equation, the cyclic basis is three-dimensional, with the closure equation given by (11). It is very important that some coefficients of the closure equation contain a parameter: from one hand, this means that the parameter k cannot be removed ('gauged out') from X by (9), since the coefficients of (11) are invariants, and, from the other hand, this has an essential influence on the structure of the class of all evolution equations which admit the linear problem (4) with this particular X , as we will see now. Let us find all those evolution equations $u_t = h(x, u, u_x, \dots, u_{x\dots x})$ which are represented by (4), where X is given by (5) and $T(x, u, u_x, \dots, u_{x\dots x})$ is an arbitrary traceless 2×2 matrix. Decomposing H in (7) over the cyclic basis as $H = pC_u + q\nabla C_u + r\nabla^2 C_u$ and using (11) for $\nabla^3 C_u$, we find

$$q = -\left(D_x + \frac{u_x}{u}\right)r, \quad p = \left(D_x^2 + D_x \frac{u_x}{u} + 4(k^2 - u^2)\right)r \quad (12)$$

and

$$h = (A + 4k^2 B)r, \quad (13)$$

where

$$A = D_x^2 u^{-1} D_x u - 4D_x u^2, \quad B = u^{-1} D_x u. \quad (14)$$

Taking into account that r can contain only finite-order derivatives of u and that h must be independent of k , we find from (13) that r is a polynomial in k and that the general expression for h is $h = R^n 0$, $n = 1, 2, 3, \dots$, where

$$R = AB^{-1} = D_x^2 - 4u^2 - 4u_x D_x^{-1} u \quad (15)$$

is the recursion operator and $D_x^{-1} 0$ must be interpreted as any constant. We have obtained a *discrete class* of evolution equations, namely, the integrable hierarchy of the modified KdV equation, whereas a zero-curvature representation without any parameter in the closure equation of its cyclic basis always leads to a *continual class* of evolution equations [1].

Now, let us consider the fake Lax pair of the modified KdV equation, found in [2]. Again, we need only the x -part of this Lax pair,

$$\psi_{xx} = \left(u^2 + ik \frac{u_x}{u} + k^2\right)\psi + \left(\frac{u_x}{u} - 2ik\right)\psi_x, \quad (16)$$

which is remarkably similar to (1). Using (3), we rewrite (16) as $\Phi_x = X\Phi$, where

$$X = \begin{pmatrix} 0 & 1 \\ u^2 + ik u_x/u + k^2 & u_x/u - 2ik \end{pmatrix}. \quad (17)$$

For this X , we compute the characteristic C_u (8), and then find that

$$\nabla C_u = 0. \quad (18)$$

Thus, in this ‘fake’ case, the cyclic basis is one-dimensional, and the closure equation (18) contains no parameters. As a result of this, any evolution equation of the form $u_t = D_x p(x, u, u_x, \dots, u_{x\dots x})$, where p is arbitrary, admits a zero-curvature representation (6) with X (17), as we can show by decomposing H in (7) over the cyclic basis as $H = pC_u$. (Note that the class of represented equations might be even wider if we take into account the *singular basis* [1], but we will not consider this possibility here.) No discrete hierarchy and no recursion operator appeared in this case: we obtained a continual class of evolution equations, as should be expected from the structure of the cyclic basis.

In conclusion, let us study the most general fake Lax pair from [2]. We rewrite its x -part (see (31a) in [2]) in the matrix form $\Phi_x = X\Phi$ with

$$X = \begin{pmatrix} 0 & 1 \\ \lambda f^2 + \eta \mu f - \eta^2 + \eta D_x f / f & \mu f - 2\eta + D_x f / f \end{pmatrix}, \quad (19)$$

where λ , η and μ are arbitrary parameters and $f(t, x, u, u_x, \dots, u_{x\dots x})$ is any function. Then we compute C_u and ∇C_u for this X (19), and find that

$$\nabla C_u = \frac{D_x(Ef)}{Ef} C_u, \quad (20)$$

where E is the Euler operator, $Ef = \partial f / \partial u - D_x(\partial f / \partial u_x) + D_x^2(\partial f / \partial u_{xx}) - \dots$. We see that the cyclic basis for this fake Lax pair is one-dimensional again. The structure of the closure equation (20) suggests that the class of evolution equations, represented by (6) with X (19), is continual: it contains at least all those evolution equations, for which f is a conserved density. Indeed, considering (1×1) -dimensional matrices X and T , for which $[X, T] = 0$, and choosing $X = f$, we see that (6) is nothing but a conservation law with a conserved density f , whereas the closure equation for this $X = f$ is exactly the same as for X (19), namely (20). Moreover, since the structure of the closure equation is invariant under the gauge transformations, the transformation (9) of X (19) with

$$S = e^{\eta x} \begin{pmatrix} 1 & 0 \\ \eta/f & 1/f \end{pmatrix}, \quad X' = \begin{pmatrix} 0 & 1 \\ \lambda & \mu \end{pmatrix} f \quad (21)$$

reveals the origin of (20).

We summarize that the difference between true and fake Lax pairs is clearly seen from the structure of their cyclic bases. Of course, this is an empirical fact, which requires further investigation.

References

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